Extensions of the results on powers of p-hyponormal operators to class wF(p,r,q) operators (Inequalities on Linear Operators and its Applications)

Ito, Masatoshi

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Extensions of the results on powers of p-hyponormal operators to class wF(p, r, q) operators

伊藤 公智 (Masatoshi Ito)

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Abstract

In this report, we shall show that inequalities

$$(T^{n+1^*}T^{n+1})^{\frac{n+p}{n+1}} \geq (T^{n^*}T^n)^{\frac{n+p}{n}} \quad \text{and} \quad (T^nT^{n^*})^{\frac{n+p}{n}} \geq (T^{n+1}T^{n+1^*})^{\frac{n+p}{n+1}}$$

for $0 < p \leq 1$ and all positive integer n hold for weaker conditions than p-hyponomality, that is, class F(p, r, q) defined by Fujii-Nakamoto or class wF(p, r, q) defined by Yang-Yuan under appropriate conditions of p, r and q.

1 Introduction

In this report, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

As an extension of hyponormal operators, i.e., $T^*T \geq TT^*$, it is well known that p-hyponormal operators for $p > 0$ are defined by $(T^*T)^p \geq (TT^*)^p$, and also an operator T is said to be p-quasihyponormal for $p > 0$ if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$. It is easily obtained that every p-hyponormal operator is q-hyponormal for $p > q > 0$ by Löwner-Heinz theorem *"$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$."*

On powers of p-hyponormal operators, Aluthge-Wang [1] showed that *"If T is a p-hyponormal operator for $0 < p \leq 1$, then T^n is $\frac{p}{n}$-hyponormal for any positive integer n."* As a more precise result than theirs, Furuta-Yanagida [8] obtained the following.

Theorem 1.A ([8]). *Let T be a p-hyponormal operator for $0 < p \leq 1$. Then*

$$(T^{n^*}T^n)^{\frac{p+1}{n}} \geq \cdots \geq (T^{2^*}T^2)^{\frac{p+1}{2}} \geq (T^*T)^{p+1},$$

$$\textit{that is,} \quad |T^n|^{\frac{2(p+1)}{n}} \geq \cdots \geq |T^2|^{p+1} \geq |T|^{2(p+1)}$$

and

$$(TT^*)^{p+1} \geq (T^2T^{2^*})^{\frac{p+1}{2}} \geq \cdots \geq (T^nT^{n^*})^{\frac{p+1}{n}},$$

$$\textit{that is,} \quad |T^*|^{2(p+1)} \geq |T^{2^*}|^{p+1} \geq \cdots \geq |T^{n^*}|^{\frac{2(p+1)}{n}}$$

hold for all positive integer n.

Recently, Gao-Yang [9] obtained the results on comparison of nth power and $(n+1)$th power of p-hyponormal operators for $0 < p \leq 1$.

Theorem 1.B ([9]). *Let T be a p-hyponormal operator for $0 < p \leq 1$. Then*

$$(T^{n+1^*}T^{n+1})^{\frac{n+p}{n+1}} \geq (T^{n^*}T^n)^{\frac{n+p}{n}}, \quad \textit{that is,} \quad |T^{n+1}|^{\frac{2(p+n)}{n+1}} \geq |T^n|^{\frac{2(p+n)}{n}}$$

and

$$(T^nT^{n^*})^{\frac{n+p}{n}} \geq (T^{n+1}T^{n+1^*})^{\frac{n+p}{n+1}}, \quad \textit{that is,} \quad |T^{n^*}|^{\frac{2(p+n)}{n}} \geq |T^{n+1^*}|^{\frac{2(p+n)}{n+1}}$$

hold for all positive integer n.

As an extension of hyponormal operators, it is also well known that invertible log-hyponormal operators are defined by $\log T^*T \geq \log TT^*$ for an invertible operator T. We remark that we treat only invertible log-hyponormal operators in this paper (see also [17]). It is easily obtained that every invertible p-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function. We note that log-hyponormality is sometimes regarded as 0-hyponormality since $\frac{X^p - I}{p} \to \log X$ as $p \to +0$ for $X > 0$. An operator T is paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in \mathcal{H}$. Ando [2] showed that every p-hyponormal operator for $p > 0$ and invertible log-hyponormal operator is paranormal. (Invertiblity of a log-hyponormal operator is not necessarily required.)

Yamazaki [18] showed that *"If T is an invertible log-hyponormal operator, then T^n is also log-hyponormal for any positive integer n,"* and also he obtained the following results.

Theorem 1.C ([18]). *Let T be an invertible log-hyponormal operator. Then*

$$(T^{n^*}T^n)^{\frac{1}{n}} \geq \cdots \geq (T^{2^*}T^2)^{\frac{1}{2}} \geq T^*T, \quad \textit{that is,} \quad |T^n|^{\frac{2}{n}} \geq \cdots \geq |T^2| \geq |T|^2$$

and

$$TT^* \geq (T^2T^{2^*})^{\frac{1}{2}} \geq \cdots \geq (T^nT^{n^*})^{\frac{1}{n}}, \quad \textit{that is,} \quad |T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^{n^*}|^{\frac{2}{n}}$$

hold for all positive integer n.

Theorem 1.D ([18]). *Let T be an invertible log-hyponormal operator. Then*

$$(T^{n+1^*}T^{n+1})^{\frac{n}{n+1}} \geq T^{n^*}T^n, \quad \textit{that is,} \quad |T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2$$

and

$$T^nT^{n^*} \geq (T^{n+1}T^{n+1^*})^{\frac{n}{n+1}}, \quad \textit{that is,} \quad |T^{n^*}|^2 \geq |T^{n+1^*}|^{\frac{2n}{n+1}}$$

hold for all positive integer n.

We remark that Theorems 1.C and 1.D correspond to Theorems 1.A and 1.B, respectively. On powers of p-hyponormal and log-hyponormal operators, related results are obtained in [7], [13], [22], [24] and so on.

On the other hand, in [6], we introduced class A defined by $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every invertible log-hyponormal operator belongs to class A and every class A operator is paranormal. We remark that class A is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms.

As we have pointed out in [14], we have the following result by combining [20, Theorem 1] and [15, Theorem 3] as a result on powers of class A operators. We remark that Theorem 1.E in case of invertible operators was shown in [11].

Theorem 1.E ([20][15][14]). *If T is a class A operator, then*

(i) $|T^{n+1}|^{\frac{2n}{n+1}} \geq |T^n|^2$ *and* $|T^{n^*}|^2 \geq |T^{n+1^*}|^{\frac{2n}{n+1}}$ *hold for all positive integer n.*

(ii) $|T^n|^{\frac{2}{n}} \geq \cdots \geq |T^2| \geq |T|^2$ *and* $|T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^{n^*}|^{\frac{2}{n}}$ *hold for all positive integer n.*

(i) (resp. (ii)) of Theorem 1.E is an extension of Theorem 1.D (resp. Theorem 1.C) since every invertible log-hyponormal operator belongs to class A.

As generalizations of class A and paranormality, Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto [3] introduced class $\mathrm{A}(p,r)$, Yamazaki-Yanagida [19] introduced absolute-(p,r)-paranormality, and Fujii-Nakamoto [4] introduced class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality as follows:

Definition.

(i) *For each $p > 0$ and $r > 0$, an operator T belongs to class $A(p,r)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}.$$

(ii) *For each $p > 0$ and $r > 0$, an operator T is absolute-(p,r)-paranormal if*

$$\||T|^p|T^*|^r x\|^r \geq \||T^*|^r x\|^{p+r}$$

for every unit vector $x \in H$.

(iii) *For each $p > 0$, $r \geq 0$ and $q > 0$, an operator T belongs to class $F(p,r,q)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}.$$

(iv) *For each $p > 0$, $r \geq 0$ and $q > 0$, an operator T is (p, r, q)-paranormal if*

$$\left\| |T|^p U |T|^r x \right\|^{\frac{1}{q}} \geq \left\| |T|^{\frac{p+r}{q}} x \right\| \tag{1.1}$$

for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T. In particular, if $r > 0$ and $q \geq 1$, then (1.1) *is equivalent to*

$$\left\| |T|^p |T^*|^r x \right\|^{\frac{1}{q}} \geq \left\| |T^*|^{\frac{p+r}{q}} x \right\|$$

for every unit vector $x \in H$ ([12]).

We remark that class $\mathrm{F}(p, r, \frac{p+r}{r})$ equals class $\mathrm{A}(p, r)$ and also class $\mathrm{F}(1, 1, 2)$ (i.e., class $\mathrm{A}(1, 1)$) equals class A. Similarly $(p, r, \frac{p+r}{r})$-paranormallty equals absolute-(p, r)-paranormality and also $(1, 1, 2)$-paranormality (i.e., absolute-$(1, 1)$-paranormality) equals paranormality.

Inclusion relations among these classes were shown in [3], [4], [12], [14], [15], [19] and so on (see also Theorems 3.A and 3.B). The following Figure 1 represents the inclusion relations among the families of class $\mathrm{F}(p, r, q)$ and (p, r, q)-paranormality.

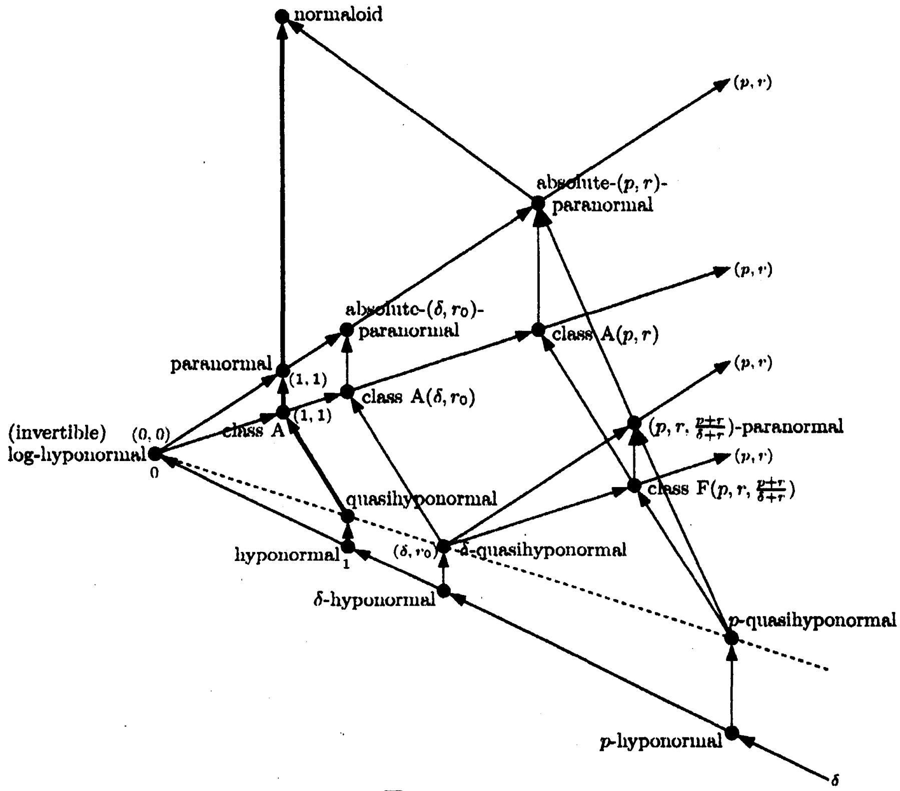

FIGURE 1

We can pick up inclusion relations among classes discussed in this report as follows: For $0 < \delta < p < 1$ and $0 < r < 1$,

$$\begin{array}{ccccc}
\delta\text{-hyponormal} & \subset & \text{class } \mathrm{F}(p,r,\frac{p+r}{\delta+r}) & \subset & \text{class } \mathrm{F}(1,1,\frac{2}{\delta+1}) \\
\cap & & \cap & & \cap \\
\text{log-hyponormal} & \subset & \text{class } \mathrm{A}(p,r) & \subset & \text{class A}
\end{array}$$

We remark that we assume invertibility on log-hyponormal operators.

In this report, as a parallel result to Theorem 1.E, we shall show that inequalities in Theorems 1.A and 1.B hold for weaker conditions than p-hyponomality, that is, class $\mathrm{F}(p,r,q)$ defined by Fujii-Nakamoto or class $\mathrm{wF}(p,r,q)$ recently defined by Yang-Yuan [23][21] (see Section 3) under appropriate conditions of p, r and q.

2 Main results

In this section, we shall show our main results.

Theorem 2.1. *If* $(|T^*||T|^2|T^*|)^{\frac{\delta+1}{2}} \geq |T^*|^{2(\delta+1)}$ *(i.e., T belongs to class $F(1,1,\frac{2}{\delta+1})$) for some* $0 \leq \delta \leq 1$*, then*

(i) $|T^{n+1}|^{\frac{2(\delta+n)}{n+1}} \geq |T^n|^{\frac{2(\delta+n)}{n}}$ *holds for all positive integer n.*

(ii) $|T^n|^{\frac{2(\delta+1)}{n}} \geq \cdots \geq |T^2|^{\delta+1} \geq |T|^{2(\delta+1)}$ *holds for all positive integer n.*

Theorem 2.2. *If* $|T|^{2(\gamma+1)} \geq (|T||T^*|^2|T|)^{\frac{\gamma+1}{2}}$ *for some* $0 \leq \gamma \leq 1$ *holds and either*

(a) $(|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2$ *(i.e., T belongs to class A) or*

(b) $N(|T|) \subseteq N(|T^*|)$

holds, then

(i) $|T^{n^*}|^{\frac{2(\gamma+n)}{n}} \geq |T^{n+1^*}|^{\frac{2(\gamma+n)}{n+1}}$ *holds for all positive integer n.*

(ii) $|T^*|^{2(\gamma+1)} \geq |T^{2^*}|^{\gamma+1} \geq \cdots \geq |T^{n^*}|^{\frac{2(\gamma+1)}{n}}$ *holds for all positive integer n.*

We need the following results in order to prove Theorems 2.1 and 2.2.

Theorem 2.A ([15]). *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$,*

(i) *If* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$*, then* $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$.

(ii) *If* $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ *and* $N(A) \subseteq N(B)$*, then* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.

Theorem 2.B ([20]). *Let A and B be positive operators. Then*

(i) *If* $(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \geq B^{\beta_0}$ *holds for fixed* $\alpha_0 > 0$ *and* $\beta_0 > 0$, *then*

$$(B^{\frac{\beta}{2}}A^{\alpha_0}B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha_0+\beta}} \geq B^{\beta}$$

holds for any $\beta \geq \beta_0$. *Moreover,*

$$A^{\frac{\alpha_0}{2}}B^{\beta_1}A^{\frac{\alpha_0}{2}} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_2}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta_2}}$$

holds for any β_1 *and* β_2 *such that* $\beta_2 \geq \beta_1 \geq \beta_0$.

(ii) *If* $A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_0}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}$ *holds for fixed* $\alpha_0 > 0$ *and* $\beta_0 > 0$, *then*

$$A^{\alpha} \geq (A^{\frac{\alpha}{2}}B^{\beta_0}A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta_0}}$$

holds for any $\alpha \geq \alpha_0$. *Moreover,*

$$(B^{\frac{\beta_0}{2}}A^{\alpha_2}B^{\frac{\beta_0}{2}})^{\frac{\alpha_1+\beta_0}{\alpha_2+\beta_0}} \geq B^{\frac{\beta_0}{2}}A^{\alpha_1}B^{\frac{\beta_0}{2}}$$

holds for any α_1 *and* α_2 *such that* $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

Lemma 2.C ([20][16]). *Let A, B and C be positive operators. Then for* $p > 0$ *and* $0 < r \leq 1$,

(i) *If* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *and* $B \geq C$, *then* $(C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{r}{p+r}} \geq C^r$.

(ii) *If* $A \geq B$, $B^r \geq (B^{\frac{r}{2}}C^pB^{\frac{r}{2}})^{\frac{r}{p+r}}$ *and* $N(A) = N(B)$, *then* $A^r \geq (A^{\frac{r}{2}}C^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$.

Lemma 2.D ([5]). *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^{\lambda} = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

We remark that Lemma 2.D holds without invertibility of A and B when $\lambda \geq 1$.

Proof of Theorem 2.1. Let $T = U|T|$ be the polar decomposition of T, and put $A_k = (T^{k^*}T^k)^{\frac{1}{k}} = |T^k|^{\frac{2}{k}}$ and $B_k = (T^kT^{k^*})^{\frac{1}{k}} = |T^{k^*}|^{\frac{2}{k}}$ for a positive integer k. We remark that $T^* = U^*|T^*|$ is also the polar decomposition of T^*.

Firstly we shall show $|T^2|^{\delta+1} \geq |T|^{2(\delta+1)}$. By the hypothesis $(|T^*||T|^2|T^*|)^{\frac{\delta+1}{2}} \geq |T^*|^{2(\delta+1)}$ for some $0 \leq \delta \leq 1$, we have

$$\begin{aligned}|T^2|^{\delta+1} &= (U^*|T^*||T|^2|T^*|U)^{\frac{\delta+1}{2}} \\ &= U^*(|T^*||T|^2|T^*|)^{\frac{\delta+1}{2}}U \\ &\geq U^*|T^*|^{2(\delta+1)}U \\ &= |T|^{2(\delta+1)}.\end{aligned}$$

Next we assume that

$$|T^{n+1}|^{\frac{2(\delta+n)}{n+1}} \geq |T^n|^{\frac{2(\delta+n)}{n}}, \quad \text{that is,} \quad A_{n+1}^{\delta+n} \geq A_n^{\delta+n} \tag{2.1}$$

holds for $n = 1, 2, \ldots, k$. By (2.1) and Löwner-Heinz theorem, we have

$$A_{k+1} \geq A_k \geq \cdots \geq A_2 \geq A_1 \tag{2.2}$$

since $\frac{1}{\delta+n} \in (0, 1]$ in (2.1). The hypothesis $(|T^*||T|^2|T^*|)^{\frac{\delta+1}{2}} \geq |T^*|^{2(\delta+1)}$ can be rewritten by $(B_1^{\frac{1}{2}}A_1B_1^{\frac{1}{2}})^{\frac{\delta+1}{2}} \geq B_1^{\delta+1}$, and also this yields $A_1 \geq (A_1^{\frac{1}{2}}B_1A_1^{\frac{1}{2}})^{\frac{1}{2}}$ by Löwner-Heinz theorem and (i) of Theorem 2.A. (2.2) and $A_1 \geq (A_1^{\frac{1}{2}}B_1A_1^{\frac{1}{2}})^{\frac{1}{2}}$ ensure

$$A_k \geq (A_k^{\frac{1}{2}}B_1A_k^{\frac{1}{2}})^{\frac{1}{2}} \tag{2.3}$$

by (ii) of Lemma 2.C since $N(A_k) = N(A_1)$ holds. We remark that $N(A_k) \subseteq N(A_1)$ holds by (2.2) and $N(A_k) = N(T^k) \supseteq N(T) = N(A_1)$ always holds. Then we get

$$A_k^k \geq (A_k^{\frac{k}{2}}B_1A_k^{\frac{k}{2}})^{\frac{k}{k+1}} \tag{2.4}$$

by (2.3) and (ii) of Theorem 2.B. Similarly, (2.2) and $A_1 \geq (A_1^{\frac{1}{2}}B_1A_1^{\frac{1}{2}})^{\frac{1}{2}}$ ensure

$$A_{k+1} \geq (A_{k+1}^{\frac{1}{2}}B_1A_{k+1}^{\frac{1}{2}})^{\frac{1}{2}}. \tag{2.5}$$

Therefore we have

$$\begin{aligned}|T^{k+1}|^{\frac{2(\delta+k+1)}{k+1}} &= (U^*|T^*||T^k|^2|T^*|U)^{\frac{\delta+k+1}{k+1}} \\ &= U^*(B_1^{\frac{1}{2}}A_k^kB_1^{\frac{1}{2}})^{\frac{\delta+k+1}{k+1}}U \\ &= U^*B_1^{\frac{1}{2}}A_k^{\frac{k}{2}}(A_k^{\frac{k}{2}}B_1A_k^{\frac{k}{2}})^{\frac{\delta}{k+1}}A_k^{\frac{k}{2}}B_1^{\frac{1}{2}}U && \text{by Lemma 2.D} \\ &\leq U^*B_1^{\frac{1}{2}}A_k^{\frac{k}{2}}A_k^{\delta}A_k^{\frac{k}{2}}B_1^{\frac{1}{2}}U && \text{by (2.4) and Löwner-Heinz theorem} \\ &= U^*B_1^{\frac{1}{2}}A_k^{\delta+k}B_1^{\frac{1}{2}}U \\ &\leq U^*B_1^{\frac{1}{2}}A_{k+1}^{\delta+k}B_1^{\frac{1}{2}}U && \text{by (2.1)} \\ &\leq U^*(B_1^{\frac{1}{2}}A_{k+1}^{k+1}B_1^{\frac{1}{2}})^{\frac{(\delta+k)+1}{(k+1)+1}}U \\ &= (U^*|T^*||T^{k+1}|^2|T^*|U)^{\frac{\delta+k+1}{k+2}} \\ &= |T^{k+2}|^{\frac{2(\delta+k+1)}{k+2}}.\end{aligned}$$

We remark that the last inequality holds by (ii) of Theorem 2.B since (2.5) holds and $k+1 \geq \delta + k \geq 1$.

Consequently the proof of (i) is complete. We can easily obtain (ii) by (i) and Löwner-Heinz theorem, so we omit its proof. □

Proof of Theorem 2.2. Let $T = U|T|$ be the polar decomposition of T, and put $A_k = (T^{k^*}T^k)^{\frac{1}{k}} = |T^k|^{\frac{2}{k}}$ and $B_k = (T^kT^{k^*})^{\frac{1}{k}} = |T^{k^*}|^{\frac{2}{k}}$ for a positive integer k. We remark that $T^* = U^*|T^*|$ is also the polar decomposition of T^*.

$|T|^{2(\gamma+1)} \geq (|T||T^*|^2|T|)^{\frac{\gamma+1}{2}}$ and condition (b) ensure condition (a) by Löwner-Heinz theorem and (ii) of Theorem 2.A, so that we have only to prove the case where condition (a) holds.

Firstly we shall show $|T^*|^{2(\gamma+1)} \geq |T^{2^*}|^{\gamma+1}$. By the hypothesis $|T|^{2(\gamma+1)} \geq (|T||T^*|^2|T|)^{\frac{\gamma+1}{2}}$ for some $0 \leq \gamma \leq 1$, we have

$$\begin{aligned} |T^{2^*}|^{\gamma+1} &= (U|T||T^*|^2|T|U^*)^{\frac{\gamma+1}{2}} \\ &= U(|T||T^*|^2|T|)^{\frac{\gamma+1}{2}}U^* \\ &\leq U|T|^{2(\gamma+1)}U^* \\ &= |T^*|^{2(\gamma+1)}. \end{aligned}$$

Next we assume that

$$|T^{n^*}|^{\frac{2(\gamma+n)}{n}} \geq |T^{n+1^*}|^{\frac{2(\gamma+n)}{n+1}}, \quad \text{that is,} \quad B_n^{\gamma+n} \geq B_{n+1}^{\gamma+n} \tag{2.6}$$

holds for $n = 1, 2, \ldots, k$. By (2.6) and Löwner-Heinz theorem, we have

$$B_1 \geq B_2 \geq \cdots \geq B_k \geq B_{k+1} \tag{2.7}$$

since $\frac{1}{\gamma+n} \in (0,1]$ in (2.6). Condition (a) can be rewritten by $(B_1^{\frac{1}{2}}A_1B_1^{\frac{1}{2}})^{\frac{1}{2}} \geq B_1$. (2.7) and $(B_1^{\frac{1}{2}}A_1B_1^{\frac{1}{2}})^{\frac{1}{2}} \geq B_1$ ensure

$$(B_k^{\frac{1}{2}}A_1B_k^{\frac{1}{2}})^{\frac{1}{2}} \geq B_k. \tag{2.8}$$

by (i) of Lemma 2.C Then we get

$$(B_k^{\frac{k}{2}}A_1B_k^{\frac{k}{2}})^{\frac{k}{k+1}} \geq B_k^k. \tag{2.9}$$

by (2.8) and (i) of Theorem 2.B. Similarly, (2.7) and $(B_1^{\frac{1}{2}}A_1B_1^{\frac{1}{2}})^{\frac{1}{2}} \geq B_1$ ensure

$$(B_{k+1}^{\frac{1}{2}}A_1B_{k+1}^{\frac{1}{2}})^{\frac{1}{2}} \geq B_{k+1}. \tag{2.10}$$

Therefore we have

$$\begin{aligned}
|T^{k+1^*}|^{\frac{2(\gamma+k+1)}{k+1}} &= (U|T||T^{k^*}|^2|T|U^*)^{\frac{\gamma+k+1}{k+1}} \\
&= U(A_1^{\frac{1}{2}} B_k^k A_1^{\frac{1}{2}})^{\frac{\gamma+k+1}{k+1}} U^* \\
&= U A_1^{\frac{1}{2}} B_k^{\frac{k}{2}} (B_k^{\frac{k}{2}} A_1 B_k^{\frac{k}{2}})^{\frac{\gamma}{k+1}} B_k^{\frac{k}{2}} A_1^{\frac{1}{2}} U^* && \text{by Lemma 2.D} \\
&\geq U A_1^{\frac{1}{2}} B_k^{\frac{k}{2}} B_k^{\gamma} B_k^{\frac{k}{2}} A_1^{\frac{1}{2}} U^* && \text{by (2.9) and Löwner-Heinz theorem} \\
&= U A_1^{\frac{1}{2}} B_k^{\gamma+k} A_1^{\frac{1}{2}} U^* \\
&\geq U A_1^{\frac{1}{2}} B_{k+1}^{\gamma+k} A_1^{\frac{1}{2}} U^* && \text{by (2.6)} \\
&\geq U(A_1^{\frac{1}{2}} B_{k+1}^{k+1} A_1^{\frac{1}{2}})^{\frac{(\gamma+k)+1}{(k+1)+1}} U^* \\
&= (U|T||T^{k+1^*}|^2|T|U^*)^{\frac{\gamma+k+1}{k+2}} \\
&= |T^{k+2^*}|^{\frac{2(\gamma+k+1)}{k+2}}.
\end{aligned}$$

We remark that the last inequality holds by (i) of Theorem 2.B since (2.10) holds and $k+1 \geq \gamma+k \geq 1$.

Consequently the proof of (i) is complete. We can easily obtain (ii) by (i) and Löwner-Heinz theorem, so we omit its proof. □

Remark. By putting $\delta = 0$ in Theorem 2.1 and $\gamma = 0$ in Theorem 2.2, we get Theorem 1.E since $(|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2$ (i.e., T belongs to class A) ensures $|T|^2 \geq (|T||T^*|^2|T|)^{\frac{1}{2}}$ by (i) of Theorem 2.A.

3 Classes $\mathbf{F}(p, r, q)$ and $\mathbf{wF}(p, r, q)$ operators

Recently, in order to continue the study of class $\mathrm{F}(p, r, q)$, Yang-Yuan [23][21] introduced class $\mathrm{wF}(p, r, q)$ operators as follows: For each $p \geq 0$, $r \geq 0$ and $q \geq 1$ with $(p, r) \neq (0, 0)$ and $(p, q) \neq (0, 1)$, an operator T belongs to class $\mathrm{wF}(p, r, q)$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}} \tag{3.1}$$

and

$$|T|^{2(p+r)(1-\frac{1}{q})} \geq (|T|^p|T^*|^{2r}|T|^p)^{1-\frac{1}{q}}, \tag{3.2}$$

denoting $(1-q^{-1})^{-1}$ by q^* when $q > 1$ because q and $(1-q^{-1})^{-1}$ are a couple of conjugate exponents. On discussions of class $\mathrm{wF}(p, r, q)$ (or class $\mathrm{F}(p, r, q)$), we frequently consider class $\mathrm{wF}(p, r, \frac{p+r}{\delta+r})$ (or class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$) by putting $q = \frac{p+r}{\delta+r}$ as follows: For $p \geq 0$, $r \geq 0$ and $-r < \delta \leq p$ with $(p, r) \neq (0, 0)$ and $(p, \delta) \neq (0, 0)$, an operator T belongs to class $\mathrm{wF}(p, r, \frac{p+r}{\delta+r})$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{\delta+r}{p+r}} \geq |T^*|^{2(\delta+r)} \tag{3.3}$$

and

$$|T|^{2(-\delta+p)} \geq (|T|^p|T^*|^{2r}|T|^p)^{\frac{-\delta+p}{p+r}}. \tag{3.4}$$

We remark that (3.1) is the definition of class $\mathrm{F}(p,r,q)$. We also remark that class $\mathrm{wF}(p,r,\frac{p+r}{r})$ equals class $\mathrm{wA}(p,r)$ defined in [10], and also it was shown in [15] that class $\mathrm{wA}(p,r)$ (i.e., class $\mathrm{wF}(p,r,\frac{p+r}{r})$) coincides with class $\mathrm{A}(p,r)$. On inclusion relations of classes $\mathrm{A}(p,r)$, $\mathrm{F}(p,r,q)$ and $\mathrm{wF}(p,r,q)$, the following results were obtained.

Theorem 3.A.

(i) *For invertible operator T, T is log-hyponormal if and only if T belongs to class $A(p,r)$ for all $p>0$ and $r>0$ ([3]).*

(ii) *If T belongs to class $A(p_0,r_0)$ for $p_0>0$, $r_0>0$, then T belongs to class $A(p,r)$ for any $p \geq p_0$ and $r \geq r_0$ ([15]).*

We note that log-hyponormality can be regarded as class $\mathrm{A}(0,0)$ by Theorem 3.A.

Theorem 3.B.

(i) *For a fixed $\delta>0$, T is δ-hyponormal if and only if T belongs to class $F(2\delta p, 2\delta r, q)$ for all $p>0$, $r\geq 0$ and $q\geq 1$ with $(1+2r)q \geq 2(p+r)$, i.e., T belongs to class $F(p,r,q)$ for all $p>0$, $r\geq 0$ and $q \geq 1$ with $(\delta+r)q \geq p+r$ ([4]).*

(ii) *For each $p>0$ and $r>0$, T is p-quasihyponormal if and only if T belongs to class $F(p,r,1)$. ([12]).*

(iii) *If T belongs to class $F(p_0,r_0,q_0)$ for $p_0>0$, $r_0\geq 0$ and $q_0\geq 1$, then T belongs to class $F(p_0,r_0,q)$ for any $q\geq q_0$ ([4]).*

(iv) *If T belongs to class $F(p_0,r_0,\frac{p_0+r_0}{\delta+r_0})$ for $p_0>0$, $r_0\geq 0$ and $0\leq\delta\leq p_0$, then T belongs to class $F(p,r,\frac{p+r}{\delta+r})$ for any $p\geq p_0$ and $r\geq r_0$ ([14]).*

(v) *If T belongs to class $F(p_0,r_0,\frac{p_0+r_0}{\delta+r_0})$ for $p_0>0$, $r_0\geq 0$ and $-r_0<\delta\leq p_0$, then T belongs to class $F(p_0,r,\frac{p_0+r}{\delta+r})$ for any $r\geq r_0$ ([12]).*

Theorem 3.C ([23]).

(i) *If T belongs to class $wF(p_0,r_0,q_0)$ for $p_0>0$, $r_0\geq 0$ and $q_0\geq 1$, then T belongs to class $wF(p_0,r_0,q)$ for any $q\geq q_0$ with $r_0q\leq p_0+r_0$.*

(ii) *If T belongs to class $wF(p_0,r_0,q_0)$ for $p_0>0$, $r_0\geq 0$, $q_0\geq 1$ and $N(T)\subseteq N(T^*)$, then T belongs to class $wF(p_0,r_0,q)$ for any q such that $q^*\geq q_0^*$ with $p_0q^*\leq p_0+r_0$.*

(iii) *If T belongs to class $wF(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ for $p_0 > 0$, $r_0 \geq 0$ and $-r < \delta \leq p_0$, then T belongs to class $wF(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.*

(iv) *If $p > 0$, $r \geq 0$, $q \geq 1$ with $rq \leq p + r$, then class $wF(p, r, q)$ coincides with class $F(p, r, q)$. In other words, if $p > 0$, $r \geq 0$, $0 \leq \delta \leq p$ and $\delta + r \neq 0$, then class $wF(p, r, \frac{p+r}{\delta+r})$ coincides with class $F(p, r, \frac{p+r}{\delta+r})$.*

In this section, firstly we shall get a relation between p-hyponormality and class $\mathrm{wF}(p, r, q)$ (or class $\mathrm{F}(p, r, q)$). We remark that Theorem 3.1 is a parallel result to (i) of Theorem 3.A.

Theorem 3.1.

(i) *For a fixed $\delta > 0$, T is δ-hyponormal (i.e., T belongs to class $F(p_0, 0, \frac{p_0}{\delta})$ for some $p_0 \geq \delta$) if and only if T belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for all $p \geq \delta$ and $r \geq 0$.*

(ii) *For a fixed $\delta < 0$, T is $(-\delta)$-hyponormal (i.e., T belongs to class $wF(0, r_0, \frac{r_0}{\delta+r_0})$ for some $r_0 > -\delta$) if and only if T belongs to class $wF(p, r, \frac{p+r}{\delta+r})$ for all $p \geq 0$ and $r > -\delta$.*

For $0 < \delta < p < 1$ and $0 < -\delta' < r < 1$, inclusion relations among class $\mathrm{wF}(p, r, q)$ and other classes can be expressed as the following diagram. We remark that we assume invertibility on log-hyponormal operators, and also $N(T) \subseteq N(T^*)$ is required in $(*)$.

$$\begin{array}{ccccc}
\delta\text{-hyponormal} & \subset & \text{class F}(p, r, \frac{p+r}{\delta+r}) & \subset & \text{class F}(1, 1, \frac{2}{\delta+1}) \\
\cap & & \cap & & \cap \\
\text{log-hyponormal} & \subset & \text{class A}(p, r) & \subset & \text{class A} \\
\cup & & \cup\,(*) & & \cup\,(*) \\
(-\delta')\text{-hyponormal} & \subset & \text{class wF}(p, r, \frac{p+r}{\delta'+r}) & \subset & \text{class wF}(1, 1, \frac{2}{\delta'+1})
\end{array}$$

Next we shall obtain the following corollaries led by Theorems 2.1 and 2.2, and also Theorems 1.A and 1.B follow from these corollaries.

Corollary 3.2. *If T belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for some $0 \leq \delta \leq 1$, $0 < p \leq 1$ and $0 \leq r \leq 1$ such that $-r < \delta \leq p$, then*

(i) *$|T^{n+1}|^{\frac{2(\delta+n)}{n+1}} \geq |T^n|^{\frac{2(\delta+n)}{n}}$ holds for all positive integer n.*

(ii) *$|T^n|^{\frac{2(\delta+1)}{n}} \geq \cdots \geq |T^2|^{\delta+1} \geq |T|^{2(\delta+1)}$ holds for all positive integer n.*

Corollary 3.3. *If T belongs to class $wF(p,r,\frac{p+r}{\delta+r})$ for some $-1 \leq \delta \leq 0$, $0 \leq p \leq 1$ and $0 \leq r \leq 1$ such that $-r < \delta < p$, and T satisfies $N(T) \subseteq N(T^*)$, then*

(i) $|T^{n^*}|^{\frac{2(-\delta+n)}{n}} \geq |T^{n+1^*}|^{\frac{2(-\delta+n)}{n+1}}$ *holds for all positive integer n.*

(ii) $|T^*|^{2(-\delta+1)} \geq |T^{2^*}|^{-\delta+1} \geq \cdots \geq |T^{n^*}|^{\frac{2(-\delta+1)}{n}}$ *holds for all positive integer n.*

We omit proofs of the results in this section.

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